Physical Limitations on Ray Oscillation Suppressors

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The question of whether it is possible to suppress ray oscillations in light waveguides is important for the design of light communications systems. With the help of Liouville's theorem of statistical mechanics it is shown that it is impossible to reduce simultaneously the amplitudes and the angles of ray oscillations if the ray originates in and returns to a region of low index of refraction. A reduction of both ray amplitudes and angles can be achieved only if the ray moves from a region of low to one of high index of refraction.

Liouville's theorem is used to derive a condition relating the output position and slope of a ray which traverses an optical transformer to its input position and slope. With $\mathbf{p}_i$, $\mathbf{x}_i$ *denoting the canonically conjugate variables of the output ray and* p_i , x_i *those of the input ray, the condition derived from Liouville's theorem states that the Jacobian of the transformation is one.*

$$\frac{\partial(\mathbf{p}_i , \mathbf{x}_i)}{\partial(p_i , x_i)} = 1 .$$

I. INTRODUCTION

Light transmission systems can be built in various ways. A continuous dielectric medium of rotational symmetry with an index of refraction which depends on the distance r from the optical axis

$$n = n(r)$$

is capable of guiding light rays if $n(r)$ decreases monotonically with increasing r. Another example is the beam-waveguide consisting of a series of lenses which refocuses the light beam periodically counteracting diffraction.

Both of these examples have one point in common — a ray which is launched off-axis into the waveguide follows an oscillatory trajectory. However, even if a light ray travels on-axis it will be forced into an oscillatory trajectory by any imperfection of the guidance medium.[1] To

keep the ray amplitudes small requires a very high precision of alignment whicb might he hard to obtain for long waveguides.

It seemed natural, therefore, to consider means of suppressing tbese ray oscillations, and if all such efforts fail, to ask for a general physical principle which says tbat sucb ray oscillation suppressors are impossihle.

The search for such a general principle is even more important as it is easy to construct models of heam waveguides which violate physical principles in suhtle ways thus seeming to lead to ray oscillation suppressors. One such system is shown in Fig. 1. Assume that we deform thin lenses as indicated in the figure and assume further that these lenses hehave just like plane thin lenses in that they hreak each ray hy an amount β_n which depends only on the radius r_n of the ray but not on the input angle.

$$\tan \beta_n = -r_n/f.$$

Making the paraxial approximation, which means replacing $\tan \beta_n$ hy β_n and $\tan \alpha_n$ hy α_n, we ohtain the ray equation

$$r_{n+1} = r_n + \alpha_n(z_{n+1} - z_n) \tag{1a}$$

$$\alpha_{n+1} = \alpha_n - \frac{r_{n+1}}{f}. \tag{1b}$$

If the lenses are warped to form parabolas,

$$z_{n+1} - z_n = d + b(r_n^2 - r_{n+1}^2). \tag{2}$$

Equations (1a) and (1h) together with (2) describe rays which, if they travel from the left to the right in Fig. 1, exhihit decreasing amplitudes. In fact, if one allows each ray to travel a sufficient distance they approach the axis arhitrarily closely.

It appears that we have invented a ray oscillation suppressor.

The object of this paper is to prove that such a device is impossible. So the question arises: What went wrong with the argument presented above? A closer examination shows that the assumption that β_n is

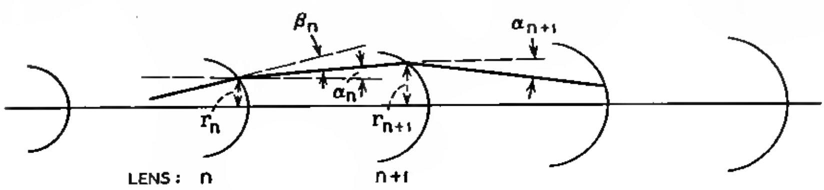

Fig. 1 — Beam-waveguide composed of warped, thin lenses.

independent of α_n violates Liouville's theorem. We will return to this question later.

The general proof of the impossibility of constructing a ray oscillation suppressor was suggested by J. R. Pierce.

II. PROOF OF THE IMPOSSIBILITY OF A RAY OSCILLATION SUPPRESSOR

The proof is based on Liouville's theorem.[2] It refers to the representation of physical systems in phase space. Phase space is the space of the canonically conjugate variables q_i and p_i describing the system. Each system is represented by one point in phase space. Many identical systems which happen to be in different states described by different values of their coordinator q_i and p_i can be described by the density of their representation points in phase space. Liouville's theorem states that the density of any given configuration of points in phase space is constant if the systems under consideration obey the canonical differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{3}$$

H is the Hamiltonian function describing the system. Another version of Liouville's theorem states that the volume containing a constant number of points in phase space remains constant in time.

For Liouville's theorem to be applicable to light rays we have only to show that light rays can be described by equations of the form (3). The derivation of the Hamiltonian equations of geometric optics can be found in Ref. 5. The derivations are sketched here for the sake of convenience.

To show this we start with Fermat's principle which states that a light ray connecting two arbitrary points P_1 and P_2 in a medium of index of refraction

$$n = n(x,y,z) \tag{5}$$

follows a path such that

$$J = \frac{1}{c}\int_{P_1}^{P_2} n\, d\, s = \text{extremum}. \tag{6}$$

Here, c is the velocity of light in vacuum and s is the path length measured along the ray trajectory. Introducing coordinates x,y,z, we can rewrite (6) as

$$J = \frac{1}{c}\int_{P_1}^{P_2} n\sqrt{1 + x'^2 + y'^2}\, dz = \text{extremum} \tag{7}$$

with

$$x' = \frac{dx}{dz} \quad \text{and} \quad y' = \frac{dy}{dz}. \tag{8}$$

Equation (7) is analogous to Hamilton's principle of least action with the Lagrangian

$$L = n\sqrt{1 + x'^2 + y'^2} \tag{9}$$

and the time t being replaced by the z-coordinate.

Once the Lagrangian of a system is known the moments p_x and p_y canonically conjugate to the x and y coordinates are defined by

$$p_x = \frac{\partial L}{\partial x'} = n\frac{x'}{\sqrt{1 + x'^2 + y'^2}} \tag{10a}$$

$$p_y = \frac{\partial L}{\partial y'} = n\frac{y'}{\sqrt{1 + x'^2 + y'^2}} \tag{10b}$$

and the Hamiltonian function by

$$H = p_x x' + p_y y' - L = -\sqrt{n^2 - p_x^2 - p_y^2}. \tag{11}$$

The variational problem (7) is solved by the equations[3]

$$x' = \frac{\partial H}{\partial p_x} \qquad y' = \frac{\partial H}{\partial p_y} \tag{12a}$$

$$p_x' = -\frac{\partial H}{\partial x} \qquad p_y' = -\frac{\partial H}{\partial y}. \tag{12b}$$

Equations (12a) and (12b) are analogous to (3) which shows that the ray description can be given in terms of canonical differential equations. The equations of (12a) are satisfied identically while the equations of (12b) lead to the well-known ray equations

$$\frac{1}{\sqrt{1 + x'^2 + y'^2}}\frac{d}{dz}\left(n\frac{x'}{\sqrt{1 + x'^2 + y'^2}}\right) = \frac{\partial n}{\partial x} \tag{13a}$$

$$\frac{1}{\sqrt{1 + x'^2 + y'^2}}\frac{d}{dz}\left(n\frac{y'}{\sqrt{1 + x'^2 + y'^2}}\right) = \frac{\partial n}{\partial y}. \tag{13b}$$

Introducing

$$\frac{ds}{dz} = \sqrt{1 + x'^2 + y'^2}, \tag{14}$$

(13a) and (13b) can be written in the more familiar form[4]

$$\frac{d}{ds}\left(n\frac{dx}{ds}\right) = \frac{\partial n}{\partial x} \tag{15a}$$

$$\frac{d}{ds}\left(n\frac{dy}{ds}\right) = \frac{\partial n}{\partial y}. \tag{15b}$$

The preceding discussion of ray dynamics was sketched only to prove that Liouville's theorem applies to light rays.

Now, we are finally in a position to prove the impossibility of a ray oscillation suppressor. To simplify the discussion let us limit the problem to two dimensions, x and z. Assume that z is the axis of the system. The phase space is now two dimensional and is spanned by the coordinates x and p_x. Let us further assume that we consider an ensemble of rays whose initial conditions are such that the representation points of all these rays fill a square area centered around the origion of phase space as shown in Fig. 2. Each ray represented in this area has a certain distance x from the optical axis z and a certain slope given by (10)

$$x' = \frac{p_x}{\sqrt{n^2 - p_x^2}}. \tag{16}$$

If an oscillation suppressor were possible we would require that all the rays initially contained in the square of phase space of Fig. 2 would approach the z-axis more closely. In addition, we would require that the angles between the rays and the z-axis don't increase or perhaps even decrease. If we look at the rays initially and finally in a region of constant index of refraction n for example in vacuum, $n = 1$, we would find that the square of Fig. 2 has deformed either into the rectangle, if the angles don't shrink, or into the smaller square, if the angles as well as the amplitudes shrink, as indicated in Fig. 2 by dotted lines. In either

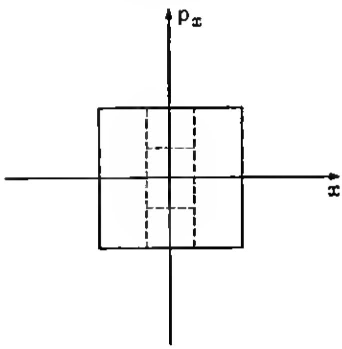

Fig. 2 — Volume in phase space occupied by light ray representation points.

case, we find that tbe area (volume in two dimensions) of phase space occupied by the points representing the initial ray positions has decreased. However, Liouville's theorem says that this is impossible so that we see that a ray oscillation suppressor is impossible. We can trade off amplitude at the expense of spread in p_x direction. In this case, either the tangent of the ray angles x' or tbe index of refraction has to increase. It is even possible to decrease both tbe ray amplitudes and angles by increasing n along the z-axis. However, tbe area in phase space has to stay constant. The initial square has to deform into a rectangle of equal area which stretches along the p_x axis. After some distance we have reached a region of high index of refraction and find both the amplitude and the ray angles decreased (but not the p_x values wbich have increased). For most applications to be able to make use of the effect, we have to leave the high index medium. But as soon as n drops to a low value the angles have to increase to keep the spread in p_x constant and again we have traded a decrease in the ray amplitudes for an increase in the ray angles. The ray position in most optical systems will eventually spread far apart if we allow the rays to travel far enough. This means that the volume in phase space, though its volume content remains constant, assumes a "filamentous" appearance and extends to many different parts of phase space.[2]

III. A BASIC RELATION FOR OPTICAL TRANSFORMERS

Liouville's theorem allows one to formulate a theorem which all rays passing through an optical device (optical transformer) have to obey.

Let us assume we have an arbitrary optical transformer with input rays whose positions and slopes are described by the canonically conjugate variables q_i, p_i and corresponding output ray with variables $\mathbf{q}_i$, $\mathbf{p}_i$. The output variables are related to the input variables by

$$\mathbf{q}_i = \mathbf{q}_i(q_i, p_i)$$
$$\mathbf{p}_i = \mathbf{p}_i(q_i, p_i).$$

The input rays may occupy a volume $dV = dq_1dq_2dp_1dp_2$ in phase space. This volume deforms, as the rays propagate, to $d\bar{V}$. Liouville's theorem states that these volumes are identical:

$$dV = d\bar{V}. \tag{17}$$

The volume $d\bar{V}$ on the right band side of (17) can be rewritten as

$$d\bar{V} = \frac{\partial(\mathbf{q}_i, \mathbf{p}_i)}{\partial(q_i, p_i)}, dq_1\, dq_2\, dp_1\, dp_2 \tag{18a}$$

or

$$d\bar{V} = \frac{\partial(\mathbf{q}_i, \mathbf{p}_i)}{\partial(q_i, p_i)} dV. \tag{18b}$$

We conclude from (17) and (18) that the Jacobian must be equal to unity

$$\frac{\partial(\mathbf{q}_i, \mathbf{p}_i)}{\partial(q_i, p_i)} = 1. \tag{19}$$

Equation (19) is stated in Ref. 6 without proof.

The derivation of (19) is based on the fact that the ray trajectory can be described by the differential equations of (13). However, there may be discontinuities in the index of refraction, n, where the ray equations can not be applied. But it is well known that rays traverse discontinuities of the index of refraction. The ray trajectory is unaltered if the discontinuity is replaced by a rapidly changing but continous transition of n. In this way we assure that the ray equations hold everywhere and that (19) is applicable even in that case.

Limiting the problem to two dimensions we can write (19) as

$$\frac{\partial \mathbf{p}}{\partial p} \frac{\partial \mathbf{x}}{\partial x} - \frac{\partial \mathbf{p}}{\partial x} \frac{\partial \mathbf{x}}{\partial p} = 1. \tag{20}$$

Equation (20) allows us to derive an interesting relation between the input and output angles of rays passing through an infinitesimally thin optical transformer (Fig. 3). If the thickness of the optical transformer shrinks to zero we have $\mathbf{x} = x$ and consequently,

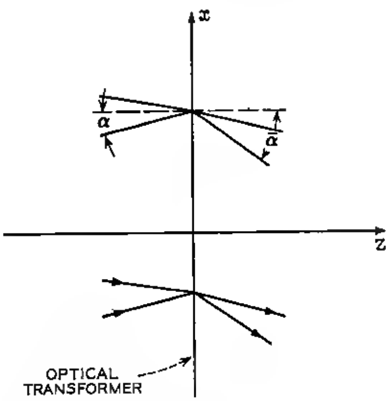

Fig. 3 — Illustrations of a thin optical transformer.

$$\frac{\partial \mathbf{x}}{\partial p} = 0 \qquad \frac{\partial \mathbf{x}}{\partial x} = 1$$

so that (20) reduces to

$$\frac{\partial \mathbf{p}}{\partial p} = 1$$

whose solution is

$$\mathbf{p} = p + f(x). \tag{21}$$

With the help of (10) we see that if

$$\frac{dx}{dz} = \tan \alpha \qquad \frac{d\mathbf{x}}{dz} = \tan \bar{\alpha},$$

it follows that

$$p = n \sin \alpha \qquad \mathbf{p} = \bar{n} \sin \bar{\alpha}$$

so that (21) can be written as

$$\bar{n} \sin \bar{\alpha} = n \sin \alpha + (\bar{n} \sin \bar{\alpha})_{\alpha=0} \, . \tag{22}$$

This is a fundamental relation which all rays passing through thin lenses or any other thin optical device have to obey.

If both $\alpha \ll 1$ and $\bar{\alpha} \ll 1$ and $\bar{n} = n = 1$, (22) simplifies

$$\beta = \bar{\alpha} - \alpha = (\bar{\alpha})_{\alpha=0} \, . \tag{23}$$

This is the relation which is used to describe the change of ray angles passing through a thin lens. Equation (22) shows that this thin lens relation holds approximately for rays which impinge nearly perpendicular to the lens. If the rays make large angles with respect to the direction normal to the lens surface (23) has to be replaced by (22). This explains the error which was made in deriving (1). If this equation is corrected by using (22) rather than (16) the ray oscillation suppressing quality of the warped thin lenses disappears.

The general expressions (19), (20), or (22) can be used to check the physical realizability of optical models.

IV. ACKNOWLEDGMENT

I am grateful to J. R. Pierce who brought Liouville's theorem to my attention, thus putting an end to attempts to invent ray oscillation suppressors.

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